MEASURING PORTFOLIO RISKS USING CONDITIONAL COPULA-AR-GARCH MODEL

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ABSTRACT

The need for predictive risk model that is able to capture complexity of stock returns is growing because of non-normality of the returns, highly inter-related financial markets and demand from regulators. This paper aims to suggest a new tool to explain dependency among different financial returns via copula, which is a distribution function that describes the dependence structure of a multi-dimension random variable, and to capture the non-normality of the returns. Elliptical and Archimedean copulas and the forecast function of the AR(1)/GARCH(1,1) model are combined, called conditional copula-AR-GARCH, to compute the Value at Risk (VaR) and empirical Expected Shortfall (ES) using full maximum likelihood estimation and Monte Carlo simulation. Estimating the portfolio's VaR and ES, comprises of Thailand Stock Index and Singapore Stock Index, the results show that the copula model captures the VaR and ES more successfully. In addition, the Gaussian copula describes the dependence structure of the portfolio return series quite well.

***Keywords**: Value at Risk, Expected Shortfall, Copula, AR-GARCH marginal, Full Maximum Likelihood*

INTRODUCTION

In financial world, changes in stock returns, interest rates, foreign exchange rates and other financial asset returns have more outliers than they would occur under normality assumption (Ruppert, 2011). Modeling financial markets data, heavy-tailed distributions are much more suitable than normal distributions. Constant variance is also another assumption that is rarely true to model financial data. For example, from January 1, 2000 to December 31, 2012, there are 16 days that the log-returns of Stock Exchange of Thailand are below -5%, and three of which are below -10%[1]. Returns on this magnitude are virtually impossible under a normal model with a constant variance even under a heavy-tailed distribution assumption, but it is more likely under a model with conditional heteroskedasticity model i.e. GARCH model. Linear correlation between assets in portfolio is regularly assumed which cannot truly capture the actual financial portfolio movement (Huang et al, 2009).

The aforementioned assumptions are widely applied as key ingredients in modeling value at risk (VaR) - the risk measurement on how the market risk of an asset or asset portfolio is likely to decrease over a certain time period under general conditions; and expected shortfall (ES) – the expected loss given a tail event, tail loss, and shortfall. This would make investors underestimate downside risks and would not be well-prepared for potential catastrophic events.

This paper aims to enhance the performance of copula-GARCH model as discussed in Huang et al (2009) by introducing the use of conditional copula-AR-GARCH model to fit the distribution of portfolio returns. The popular AR(1) and GARCH (1,1) are used to capture movement in each financial returns while four well-known copula functions are applied to build the joint distribution of the returns, namely Gaussian copula, Student-t copula, Frank copula and Clayton copula. The joint distribution will be used to find risk measures i.e. VaR and ES to see whether they can accurately capture extreme downside risks in order for financial institutions, regulators or even individual investors to efficiently manage their portfolio risks.

MODEL FOR THE MARGINAL DISTRIBUTION

Autoregressive model (AR model) and Generalized Autoregressive Conditional Heteroskedasticity model (GARCH model) are applied to capture the movement in the first and the second moment of the portfolio returns. AR model allows the flexibility in the expected return to correlate with the past returns while the GARCH model allows the volatility to be clustered and respond to shocks in previous periods (Ruppert, 2011). The marginal model is built on the classical AR-GARCH model in which the error term is to follow the normal distribution and Student-t distribution respectively.

Let the returns of an asset be given by $\{Y_t\}; t = 1, \dots, T$. AR(1)-GARCH(1,1) with an error term is a standard normal distribution (AR-GARCH-n) or a standardized Student-t (AR-GARCH-t) distribution respectively, where the model is following:

$$\begin{aligned} Y_t &= \mu_t + a_t \\ \mu_t &= \varphi + \emptyset\mu_{t-1} \\ a_t &= \sigma_t\varepsilon_t \\ \sigma_t^2 &= \alpha_0 + \alpha_1 a_{t-1}^2 + \beta\sigma_{t-1}^2 \\ \varepsilon_t &\sim N(0,1) \text{ or } \varepsilon_t \sim t_d \end{aligned} \qquad (1)$$

[1] Data are obtained from Bloomberg terminal as of February 13th 2013.

Here, $\mu_t = E(Y_t|\Omega_{t-1}) = E(\mu_t|\Omega_{t-1})$ is the unconditional mean of the asset returns, $\sigma_t^2 = Var(Y_t|\Omega_{t-1}) = Var((a_t|\Omega_{t-1})$ is the conditional variance, $\emptyset < 1$ is assumed to ensure stationary process in the first moment; φ is assumed to be zero for simplicity and reduce computational time, $\alpha_0 > 0$, $\alpha_1 \geq 0$, $\beta \geq 0$ and $\alpha_1 + \beta < 1$ are also assumed to ensure stationary process in the second moment; and Ω_{t-1} is the information set at time t-1 and d is the degree of freedom for the Student-t distribution.

The full maximum likelihood estimation method ("full MLE") will be applied in this case. To construct the likelihood function, let $\theta_{t-1} = \{Y_{t-1}, \ldots, Y_0\}$ be the information set known at time t-1, the joint marginal distribution of the series of the returns is $f(Y_0, \ldots, Y_t) = f(Y_t|\Omega_{t-1})f(Y_{t-1}|\Omega_{t-2}) \ldots f(Y_1|\Omega_0)f(Y_0)$ and the log-likelihood function ("LLF") can be written as follow:

$$LLF = \sum_{k=0}^{n-1} f(a_{n-k}|\Omega_{n-k-1}) \quad (2)$$

This can be estimated using the model expected value and volatility equation for any assumed distribution for ε_t. Here, LLF can be maximized numerically to obtain MLE. We use the observation $(y_0, \ldots, y_t)$ to obtain the conditional marginal distribution of Y_{t+1} defined as follows:

$$P(Y_{t+1} \leq y|\Omega_t) = P\left(\varepsilon_{t+1} \leq \frac{(y-\varphi-\phi u_t)}{\sqrt{\alpha_0+\alpha_1 a_t^2+\beta\sigma_t^2}}\middle|\Omega_t\right) = \begin{cases} N\left(\frac{(y-\varphi-\phi u_t)}{\sqrt{\alpha_0+\alpha_1 a_t^2+\beta\sigma_t^2}}\middle|\Omega_t\right), \text{if } \varepsilon_t \sim N(0,1) \\ t_d\left(\frac{(y-\varphi-\phi u_t)}{\sqrt{\alpha_0+\alpha_1 a_t^2+\beta\sigma_t^2}}\middle|\Omega_t\right), \text{if } \varepsilon_t \sim t_d \end{cases} \quad (3)$$

COPULA THEORY AND ESTIMATION PROCEDURE

Sklar's Theorem

Copulas are a popular method for modelling multivariate distributions. A copula models the dependence between the variates in a multivariate distribution and can be combined with any set of univariate distributions for the marginal distributions. The use of copulas allows us to take advantage of the wide variety of univariate models that are available. A copula is a multivariate CDF whose univariate marginal distributions are all Uniform(0,1). Suppose that a vector $\mathbf{Y} = (Y_{i,1}, \ldots, Y_{i,d})$ has a multivariate CDF F_Y with a continuous marginal univariate CDFs $F_{Y_1}, \ldots, F_{Y_d}$, then each of $F_{Y_1}(Y_1), \ldots, F_{Y_d}(Y_d)$ is Uniform (0,1) distributed. Therefore, the CDF of $\{F_{Y_1}(Y_1), \ldots, F_{Y_d}(Y_d)\}$ is a copula. This CDF is called the copula of Y and denoted by C_Y. C_Y contains all information about dependencies among the components of Y but has no information about the marginal CDFs of Y. Since C_Y is the CDF of $\{F_{Y_1}(Y_1), \ldots, F_{Y_d}(Y_d)\}$, by the definition of a CDF we have:

$$\begin{aligned} C_Y(u_1, \ldots, u_d) &= P\{F_{Y_1}(Y_1) \leq u_1, \ldots, F_{Y_d}(Y_d) \leq u_d\} \\ &= P\{Y_1 \leq F_{Y_1}^{-1}(u_1), \ldots, Y_d \leq F_{Y_d}^{-1}(u_d)\} \end{aligned}$$

Let $u_j = F_{Y_j}(Y_j)$, j = 1,…,d, we get:

$$F_Y(y_1, \ldots, y_d) = C_Y\{F_{Y_1}(y_1), \ldots, F_{Y_d}(y_d)\} \quad (4)$$

Equation (4) is part of a famous Sklar's theorem which states that the F_Y can be decomposed into copula C_Y which contains all information about the dependencies among $(Y_1, \ldots, Y_d)$ and the univariate marginal CDFs $F_{Y_1}(Y_1), \ldots, F_{Y_d}(Y_d)$ which contain all information about the univariate marginal distributions.

If we apply chain rule on (4), we get:

$$f_Y(y_1, \ldots, y_d) = c_Y\{F_{Y_1}(y_1), \ldots, F_{Y_d}(y_d)\} f_{Y_1}(y_1) \cdots f_{Y_d}(y_d) \quad (5)$$

where $c_Y = \frac{\partial^d}{\partial u_1 \cdots \partial u_d} C_Y$

The Copula Family

The copula family used in this paper includes commonly used copulas which are the elliptical copulas i.e. Gaussian copula and Student-t copula and Archimedean copulas i.e. Frank copula and Clayton copula. The Archimedean copula allows modelling a big variety of different dependence structures. The CDF of each copula family is defined below. For notational convenience, we denote $F_{Y_i}(Y_i) = u_i$

- ***Gaussian copula***

$$C_{Gaussian}(u_1, u_2; \rho) = \Phi_\rho\left(\Phi^{-1}(u_1), \Phi^{-1}(u_2)\right) \quad (6)$$

where Φ is the CDF of a standard normal distribution and ρ is a correlation factor

- ***Student-t copula***

$$C_T(u_1, u_2; d^*, \rho) = t_{d^*,\rho}(t_{d_1}^{-1}(u_1), t_{d_2}^{-1}(u_2)) \quad (7)$$

where t_d is a standardized Student-t distribution with d degree of freedom and ρ is a correlation factor

- ***Frank copula***

$$C_{Frank}(u_1, u_2; \lambda) = \frac{1}{\lambda}\log\left(\frac{\lambda(1-e^{-\lambda})-(1-e^{-\lambda u_1})(1-e^{-\lambda u_2})}{(1-e^{-\lambda})}\right) \tag{8}$$

where $\lambda \in (-\infty, 0) \cup (0, \infty)$ represents dependence between two random variables

- ***Clayton copula***

$$C_{Clayton}(u_1, u_2; \omega) = (u_1^{-\omega} + u_2^{-\omega} - 1)^{\frac{-1}{\omega}} \tag{9}$$

where $\omega \in [1, \infty)$ represents dependence between two random variables

Calibrating copula

This paper applies the full MLE to calibrate the model. Assume that we have an i.i.d. sample $Y_i = (Y_{i,1}, \ldots, Y_{i,d}), i = 1, \ldots, n$ and we wish to calibrate the copula of Y_i and its marginal distributions. Suppose we have parametric models for the marginal CDFs as well as the parametric model for the copula density. We can obtain the log-likelihood as follows:

$$\begin{aligned} LLF(\theta_1, \ldots, \theta_d; \theta_c) &= \sum_{i=1}^{n}(\log[c_Y\{F_{Y_1}(Y_{i,1}|\theta_1), \ldots, F_{Y_d}(Y_{i,d}|\theta_d)|\theta_c\}] + \log\{f_{Y_1}(Y_{i,1}|\theta_1)\} + \cdots \\ &\quad + \log\{f_{Y_d}(Y_{i,d}|\theta_d)\}) \end{aligned} \tag{10}$$

The maximum likelihood estimation finds the maximum of $LLF(\theta_1, \ldots, \theta_d; \theta_c)$ over the entire set of parameters $(\theta_1, \ldots, \theta_d; \theta_c)$. Given a set of marginal distributions and a copula, the previous log-likelihood may be written, and by maximization we obtain the maximum likelihood estimator:

$$\hat{\theta}_{MLE} = \text{argmax } LLF(\theta) \tag{11}$$

Estimation Of Portfolio's Value At Risk And Expected Shortfall

If $X_{p,t}$ is the loss over the holding period T, then $VaR(\alpha)$ is the αth upper quantile of L. For any continuous loss distribution:

$$P\{X_{p,t} < VaR_t(\alpha)\} = P\{X_{p,t} \leq VaR_t(\alpha)\} = \alpha \tag{12}$$

However, VaR discourages diversification (Ruppert, 2011) and for this reason it is replaced by newer risk measures e.g. expected shortfall (ES) which is expected loss given that the loss exceeds VaR. For any continuous loss distribution:

$$ES(\alpha) = E\{X_{p,t}|X_{p,t} \leq VaR_t(\alpha)\} \tag{13}$$

We consider the portfolio return $X_{p,t}$ composed by two asset returns denoted by $X_{1,t}$ and $X_{2,t}$ respectively. The portfolio return is approximately equal to the following:

$$X_{p,t} = wX_{1,t} + (1-w)X_{2,t} \tag{14}$$

where w and (1-w) are the portfolio weights of asset 1 and asset 2
The portfolio VaR can be derived as follows:

$$\begin{aligned} P(X_{p,t} \leq VaR_t(\alpha)|\Omega_{t-1}) &= P(wX_{1,t} + (1-w)X_{2,t} \leq VaR_t(\alpha)|\Omega_{t-1}) \\ &= P(X_{1,t} \leq 2VaR_t(\alpha) - X_{2,t}|\Omega_{t-1}) = 0.05 = \alpha \end{aligned} \tag{15}$$

In this paper, the portfolio is arbitrarily assumed equally weighted (w = 0.5) but this is not a constraint and it can be varied freely; where α is assumed 0.05 for 95% confidence level.
Since we have derived the continuous joint distribution of the portfolio returns, we can compute the VaR using the following equation:

$$\begin{aligned} P(X_{p,t} \leq VaR_t(\alpha)|\Omega_{t-1}) &= \int_{-\infty}^{\infty}\int_{-\infty}^{2VaR_t - x_{2,t}} f(x_{1,t}, x_{2,t}|\Omega_{t-1})\, dx_{1,t}dx_{2,t} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{2VaR_t - x_{2,t}} c(F(x_{1,t}), F(x_{2,t})|\Omega_{t-1}) f(x_{1,t}|\Omega_{t-1}) f(x_{2,t}|\Omega_{t-1})\, dx_{1,t}dx_{2,t} \end{aligned} \tag{16}$$

ES can also be computed as follows:

$$\begin{aligned} E(X_{p,t}|X_{p,t} \leq VaR_t(\alpha); \Omega_{t-1}) = &\int_{-\infty}^{\infty}\int_{-\infty}^{2VaR_t - x_{2,t}} x_{p,t}\, c(F(x_{1,t}), F(x_{2,t})|\Omega_{t-1}) \ldots \\ &f(x_{1,t}|\Omega_{t-1}) f(x_{2,t}|\Omega_{t-1})\, dx_{1,t}dx_{2,t};\ X_{p,t} = \frac{X_{1,t}}{2} + \frac{X_{2,t}}{2} \end{aligned} \tag{17}$$

EMPIRICAL RESULTS

Data And The Portfolio Returns

This paper aims at examining the performance of the conditional copula-AR-GARCH model for the period between January 5, 2000 and February 13, 2012 with 3031 daily observations of Thailand's SET index ("SET") and Singapore's Strait Times Index ("ST"). The data are obtained from Bloomberg Terminal database. To exactly match the returns[2] between the two markets, holidays are eliminated. The market returns and absolute returns of SET and ST are illustrated in Figure 1.

The stock returns swing around zero and there is a strong evidence of heteroskedasticity and volatility clustering in the financial returns. Negative skewness (negative returns are more likely) and excess kurtosis (extreme returns are more frequent than normal distribution would predict) are also observed.

Figure 1: Daily returns and absolute returns of SET and ST

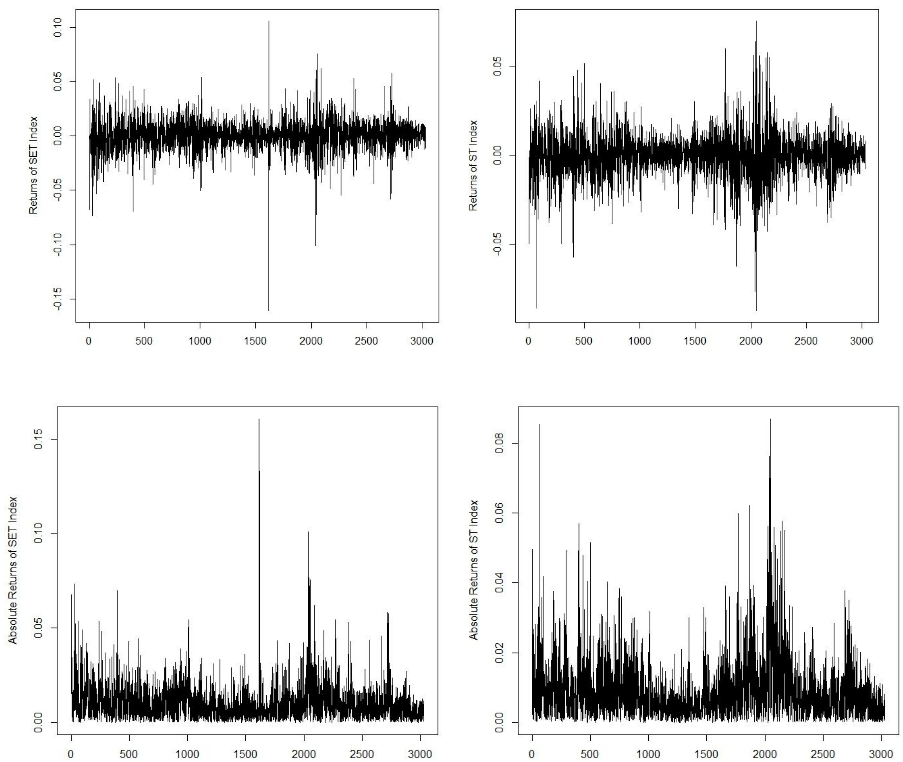


Table 1: Summary of Descriptive Statistics

	SET	ST
Sample size	3031	3031
Mean	0.000302	-0.000044
Standard deviation	0.014532	0.012725
Skewness	-0.669683	-0.245133
Excess kurtosis	8.679657	4.662093

Parameters And Goodness Of Fit Measures

The data set are separated into two groups i.e. sample-in and sample-out data, in order to test whether the VaR and ES estimates are adequate. The sample-in data contain the first 2531 observations and the remaining 500 observations are sample-out data for the test. All the marginal model distributions and copula functions are estimated using the sample-in data containing 2531 observations. The copula-AR-GARCH modeling results are shown in table 2. Using AIC and IC criteria for model selection, it can be seen that the best fitting model is the Clayton copula with normal marginal distributions which has the smallest AIC and BIC.

[2] The returns are defined as $\ln\frac{P_t}{P_{t-1}}$ where P_t is the value of the index at time t

Table 2: Summary of parameter estimates for marginal distributions, copula families and model selection statistics

Copula family	Parameters	AR-GARCH-n		AR-GARCH-t	
		SET	ST	SET	ST
Gaussian copula	θ	0.000203	0.000001	0.044521	0.000001
	α_0	0.002922	0.001001	0.001857	0.001686
	α_1	0.014908	0.015034	0.015932	0.012330
	β	0.333659	0.332811	0.254238	0.218857
	d	n/a		23.16277	23.52564
	ρ	0.979000		0.971841	
	LLF	11617.26		12321.96	
	AIC	23216.52		24621.92	
	BIC	23203.89		24606.48	
Student-t copula	θ	0.000205	0.000001	0.046700	0.000001
	α_0	0.003757	0.001001	0.002488	0.001001
	α_1	0.019401	0.019491	0.014797	0.014879
	β	0.367837	0.366990	0.317199	0.315482
	d	n/a		21.11598	21.58748
	d^*	2.807232		2.643279	
	ρ	0.955618		0.98	
	LLF	11871.95		12122.60	
	AIC	23723.90		24221.20	
	BIC	23709.86		24204.36	
Frank copula	θ	0.899260	0.899954	0.029669	0.000897
	α_0	0.001429	0.001278	0.000726	0.000726
	α_1	0.037719	0.037641	0.019482	0.019719
	β	0.681615	0.678962	0.363259	0.362954
	d	n/a		18.53323	19.14135
	ρ	25.310600		0.781442	
	LLF	11773.08		12022.95	
	AIC	23528.1600		24023.90	
	BIC	23515.5304		24008.46	
Clayton copula	θ	0.03386702	0.000001	0.004403	0.000001
	α_0	0.001001	0.0018976	0.001816	0.000636
	α_1	0.04847109	0.04838266	0.048052	0.048040
	β	0.8851598	0.857781	0.693982	0.663313
	d	n/a		3.257374	4.675744
	ρ	38.981220		4.155002	
	LLF	8456.60		10960.26	
	AIC	16895.2080		21898.52	
	BIC	16882.5784		21883.08	

Results from value at risk and expected shortfall

Using parameter estimated obtained from the sample-in data; the first forecast for day 2532 is obtained. Then using the data for day 1-2532 to compute the forecast for day 2533 until the sample-out data are used up. By applying equation 16 and 17, the forecast VaR and ES are obtained and compared against the realized portfolio returns. If the realized returns exceed VaR or ES, they are recorded as 'violations' as summarized in Table 3 below. Gaussian copula with Student-t marginal distributions outperforms other models by having lowest number of violations.

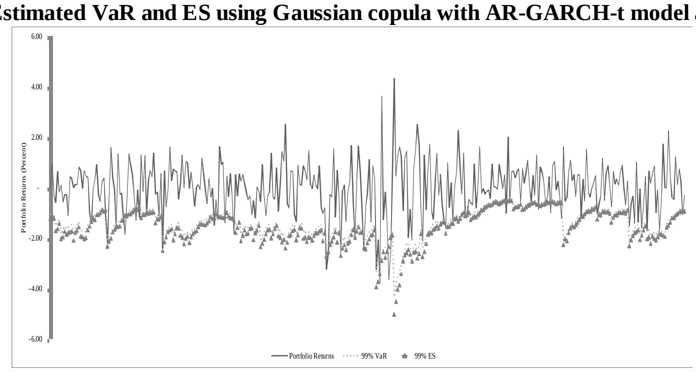
Table 3: Number of violations of the VaR and ES estimation using $\alpha = 0.05$ out of 500 trading days

Copula	VaR		ES	
	AR-GARCH-n	AR-GARCH-t	AR-GARCH-n	AR-GARCH-t
Gaussian	25	25	23	21
Student-t	27	36	26	32
Frank	22	28	20	25
Clayton	28	41	26	36

Table 4: Number of violations of the VaR and ES estimation using $\alpha = 0.01$ out of 500 trading days

Copula	VaR		ES	
	AR-GARCH-n	**AR-GARCH-t**	**AR-GARCH-n**	**AR-GARCH-t**
Gaussian	17	13	15	12
Student-t	17	19	15	18
Frank	18	19	16	17
Clayton	16	20	15	19

Figure 2: Estimated VaR and ES using Gaussian copula with AR-GARCH-t model at $\alpha = 0.01$



CONCLUSION AND FURTHER STUDY

So far it is observed that in modelling return distribution of the portfolio consisting 50% SET and 50% ST, the Frank copula with Student-t distribution best describes the return distribution of the portfolio; however, the Gaussian copula with AR-GARCH-t marginal distribution model outperforms other models in forecasting VaR and ES.

This study can be generalized in many possible directions. One of which is incorporating skewness factor in the marginal distribution. As suggested in Table 1, the returns are negatively skewed. This copula-AR-GARCH model could be developed and better capture the downside risks. Stochastic dependence structure can be included as well since there are many literatures (see Ruppert (2011) for example) suggest that correlation between assets is higher during the financial crisis.

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